Doing maths like a research mathematician

MATHSCRAFT CURRICULUM

http://curriculum.mathscraft.org

GENERAL OVERVIEW

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http://acems.org.au/

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BACKGROUND

Mathematics is a set of connected *ideas* that has enabled humans to solve many problems and create many things of value to our lives.

Learning mathematics is a process of becoming familiar and operational with *mathematical ideas*.

Doing mathematics, by contrast, is a process where *ideas that come to mind* are sequenced in unfamiliar chains, in order to *explore, solve, create and understand* the previously unexplored, previously unsolved, yet to be created and not yet understood. *Doing mathematics* is an adventure, often quite an exciting adventure.

Ideas, in this context, refer to *facts, concepts, processes and/or associations*.

The value of mathematics to our lives is not in question. What remains in question, after decades of effort, is how to positively engage school-aged people with mathematics, so they have the best chance of *doing mathematics*, and thereby positively influencing their own lives and the lives of those around them.

Despite the introduction of many alternative ways of engaging students with mathematics at school, over many decades, the number of students studying higher levels of mathematics continues to fall and the claims of widespread disengagement continue.

Most people who *do mathematics* will tell you it is engaging, wonderfully satisfying, challenging and fulfilling, but not easy. As such, it seems sensible that if school students were engaged in *doing mathematics*, while continuing to learn new skills and processes at a necessary rate, we might be part way home to quelling the fall in numbers and the disengagement. But how?

Doing mathematics is made much harder if the doer does not readily *recall* the ideas they need to recall or, if they can recall them, are then unsure about how to *use* them. In other words, *doing mathematics* is hard for an individual if they are not *operational* with the facts, concepts, processes and associations that are required.

So, could positive engagement be as simple as ensuring students are operational with the required ideas when *doing mathematics*? That is, engaging students with the:

- previously *unexplored* (by them),
- previously *unsolved* (by them),
- yet to be *created* (by them), and
- not yet *understood* (by them),

but being as sure as possible that the student will be able to *recognise* the need for an idea, and then *use* it with confidence.

Such a situation is not dissimilar to the way research mathematicians work. Research mathematicians use mathematics with which **they** are operational, to work on problems that are previously unsolved and not yet understood **by anyone**. The solving of a problem often leads to new ideas and new problems resulting in what we term mathematical *adventures*.

Imagine if students had the opportunity to *do mathematics* like a research mathematician, in areas that have been unexplored **by them***.* This is exactly what MathsCraft is all about. There are an amazing number of simply stated problems that lead to delightful *adventures* that only require ideas with which students are already operational.

MathsCraft: Doing Maths like a Research Mathematician is a program that offers all students the opportunity to use *facts, concepts, processes and associations* with which they are operational, in authentic, engaging and challenging adventures, that develop their ability to explore, solve, create and understand. In doing so, the joy of *doing mathematics* is nurtured and our experience suggests that it drives interest in, and provides a purpose for, learning new mathematical knowledge and skills.

While a problem is the doorway to a MathsCraft adventure, MathsCraft is not just a set of problems. *Doing mathematics* requires the doer to behave in certain ways and MathsCraft allows students to learn and practice these critical behaviours. Since *doing mathematics* requires facts, concepts, processes and associations with which the doer is already operational, MathsCraft is not an alternative way to learn new skills and knowledge. Therefore, not all the time devoted to school mathematics should be devoted to MathsCraft; a considerable amount needs to be devoted to ensuring students learn the facts, concepts, processes and associations they will need to *do mathematics* in the future.

One way to incorporate MathsCraft into school mathematics is through the MathsCraft Curriculum.

BEING OPERATIONAL: A KEY PRINCIPLE

For someone to be *operational with* a fact, concept, process or association means two things:

- it can come to mind prompted only by the problem being considered, and independently of it being recently taught, **and**
- the person is confident in its use and can use it to good effect.

Becoming operational with a fact, concept, process or association requires experience; that is, practice and time. The required amount of practice and time varies from person to person. It can be over a period of years.

Our experience suggests that if, in order to solve a problem, students require facts, concepts, processes or associations to come to mind that have only recently been taught, then only a small proportion of students will positively engage. However, if sufficient time has passed since these ideas were first taught, more students will be operational with them, and a higher proportion of the students will be able to naturally engage with problems that require their recall.

A MathsCraft adventure is considered appropriate for a particular student if they are operational with the facts, concepts, processes and associations required. For many students, this is most likely if these ideas were first learned some time before they engage with the adventure.

This time-lag, between the learning of facts, concepts, processes and associations and needing them to do mathematics, gives more students the chance to find *doing mathematics* engaging, wonderfully satisfying, challenging and fulfilling (but not easy).

See also Time-lag, p.15.

- MathsCraft is **not** just a set of problems.
- MathsCraft is a way to engage students with *doing mathematics*, using *facts, concepts, processes and associations* with which they are already *operational*.
- Students start with a *problem*, which results in an *adventure* during which they *explore, solve, create and understand*.
- MathsCraft models a set of *behaviours* and provides the opportunity to develop those behaviours.
- MathsCraft is **not** an alternative way of learning new skills and knowledge.
- **Only some** of the time devoted to school mathematics should be allocated to MathsCraft. A considerable amount of time should be allocated to learning *facts, concepts, processes and associations*.
- One way to incorporate MathsCraft into school mathematics is through the MathsCraft Curriculum.

OVERVIEW

The MathsCraft Curriculum is a companion curriculum to the **Australian Curriculum: Mathematics**, and hence all local implementations. It is a structured program that supports teachers to successfully engage students in doing mathematics.

The MathsCraft Curriculum is designed for all students in Years 5–10. It requires a minimum of 6–10 hours of class time[1].

The Curriculum includes:

- A carefully curated set of problems to present to a whole class of students, working in groups of 2 or 3
- Support resources, including:
  - Videos accompanying the problems, demonstrating how the adventure might unfold
  - A written document for each problem, demonstrating how the adventure might unfold
- A List of Mathematical Behaviours to help teachers to mentor-by-modelling and for use with assessment (see Appendix B, p.20)

To access the MathsCraft Curriculum, go to http://curriculum.mathscraft.org.

EXPECTED STUDENT OUTCOMES

The students will:

- Explore mathematics
- Have independent ideas that help them explore, solve, create and understand
- Share ideas and collaborate mathematically
- Develop an appreciation for logical arguments
- Learn to build logical arguments
- Learn to use reasoning to become sure of an idea, rather than relying on an authority
- Learn to write down what they are thinking
- Appreciate "dead ends" as a positive step in their adventure
- Develop their critical and creative thinking
- Develop their resilience and persistence

[1] The minimum required time depends on the problems chosen – see Time Commitment, p.8.

FEEDBACK FROM TEACHERS IN THE 2019 PILOT

[It] was exciting to see that once they realised it was about offering suggestions/ observations and questions how the level of interaction increased. Their usual teacher commented on how she liked the process of testing, refining and moving forward.

The [students] have developed resilience and risk taking and the idea that there may not be a final, and only one, answer. What a great project.

My students are doing a compulsory investigation now and I am already seeing the benefits of the MathsCraft programme — they aren't complaining about being lost so much, and they are careful to see that a pattern ***always*** *holds, not just for a few cases — MathsCraft fruits!*

[Sent one month after the completion of the curriculum]

[We] could not bring ourselves to clean the board. The girls left with a real buzz and we could hear them still saying: "but what if …"

I loved it, despite the fact that I had a really difficult class ... [with a] generally negative attitude to mathematics ... I really liked the growth that I saw in the students over the sessions — they learned to collect data and make mathematical observations, as well as embrace the "dead ends" as being valuable things to learn from. They rose to the challenges I gave them many times, and were generally engaged with the process.

Teachers reported increased enjoyment in their teaching by spending some class time *doing mathematics*.

STRUCTURE OF THE MATHSCRAFT CURRICULUM

CLASS SETUP

Students work in groups. Three is the preferred group size. Where possible, students work in the same group for the entirety of the curriculum.

In general, the required materials are:

- Pen/pencil and paper for the students to work with;
- One or more large whiteboards at the front of the room.

PROBLEMS

The curriculum contains a carefully curated set of problems.

Problems come in different types: “Smaller” problems have a narrower scope of adventure than “Larger” problems. “Flexible” problems can be narrowed or expanded as desired.

During two terms, at least six problems from the problem set are presented. If desired, one or more of these can be used as an assessment problem, see Assessment, p.12.

Each problem comes with support resources that demonstrate how the adventure might unfold, including a video and a written document.

TIME COMMITMENT

The MathsCraft Curriculum is designed to be implemented over two consecutive school terms, on a regular basis; e.g., one 40–60 minute MathsCraft session per fortnight.

It requires a minimum of 6–10 hours of class time in total, depending on which problems are used.

This class time is used to introduce at least six MathsCraft problems, that will each spark an adventure. Of course, a teacher can choose to spend more than the specified time providing opportunities for MathsCraft adventures.

Each problem should be allocated a total time of approximately:

- 60 minutes for Smaller problems,
- 100 minutes for Larger problems,
- 60–100 minutes as desired for Flexible problems.

Depending on how the adventure sparked by a problem goes, the allocated time may be slightly extended or reduced.

An adventure sparked by one problem can take place over multiple sessions.

MATHEMATICAL BEHAVIOURS

There are some things a mathematician might do when faced with an interesting problem to ponder and have ideas about. Appendix B (p.20) contains a detailed (though by no means complete) description of such mathematical behaviours.

These behaviours are, in part, the content of the MathsCraft Curriculum. The description of these behaviours helps teachers to mentor-by-modelling and are used for assessment.

A brief summary of these behaviours is:

- Creating data[2] that become the stimuli for ideas
- Having an idea (or two or three)
- Pursuing the idea to test its worth:
  - Rejecting that idea
  - Modifying that idea
  - Working to become sure, beyond all doubt
- Creating a new problem that builds from the one just studied

One might think about these as four phases of a MathsCraft adventure:

1. calculative phase
2. inductive phase
3. deductive phase and
4. extension phase

While these are numbered, a mathematician's adventure may jump around, quite a lot, between these phases. Often, the adventure is anything but linear.

Some of these behaviours will occur naturally, and some come by being modelled and practised.

[2] We use data here to mean anything that can be used to stimulate an idea. This could be a set of examples, cases, diagrams, calculations, etc.

MATHSCRAFT SESSIONS

JOB OF THE TEACHER

The teacher's job is to present the problem and assist the flow of the adventure.

Assisting the flow of the adventure, in this curriculum, requires you to do as little as possible, but as much as needed, to create the environment in which students have the chance to *have an idea*.

You can do almost anything except have the idea for them.

Problems can remain unsolved, and questions that arise can remain unanswered, but you may wish to reach some satisfying points before finishing the session.

An example of how a session might be run:

- you present the problem
- groups work for a time
- you pick some groups' ideas for them to share, to act as further stimulus for the others, or to connect to other ideas
- groups work a bit more
- etc.

The video and written documents accompanying each problem can provide some scaffolding for the teacher to assist the flow of the adventure.

JOB OF THE STUDENT GROUP

The job of the student group is to work together on the problem, have ideas and pursue them, think of questions to ask, and generally try to get to the heart of what is going on — that is, *do mathematics*.

Students may be called upon to share an idea with the rest of the class.

As the groups work on each problem, they should keep a written account of their adventure.

WRITTEN ACCOUNTS

Their written account[3] should not be a polished solution, but rather a "live recording" of their adventure. This should include:

- what they're thinking and doing
- what they try, what fails, what works
- patterns they see, connections they make
- questions that occur to them
- things that puzzle them
- discoveries made

[3] Writing down their thinking as they go is a skill that is separate to mathematical skills. Students will need to practise this during the classes.

If they try something that doesn't work, they should not erase it or cross it out, but simply indicate that it didn't work. Such “dead ends” are often as valuable to the end-goal as the great moments themselves.

The written account does not need to be grammatically correct nor mathematically perfect, but it should be clear enough to be understood by a maths teacher who wasn’t present at the time.

If an adventure takes place over multiple sessions, written accounts should be collected by the teacher at the end of a session and given back at the start of the next session.

Various forms of written accounts are acceptable[4].

[4] Samples of student work can be found under https://curriculum.mathscraft.org/assessment (registration required).

ASSESSMENT

Assessment is optional. This section provides a guide for those who choose to assess.

OPTIONS

The preferred method of assessment is to choose a grade for a whole group, based on evidence of their behaviours during the last of the problems presented.

However, if required, you may instead choose to assess students individually, and/or assess based on the students' behaviours in all the problems.

GRADES

The assessment in the MathsCraft Curriculum is not based on mathematical *results*, but on mathematical *behaviours*.

The teacher assesses the work of a group by observing and collecting evidence of how the group behaves mathematically.

Use the List of Mathematical Behaviours in Appendix B (p.20) to decide on a grade for each group. The list should be treated as a guide. Not all behaviours are needed for every problem, and some worthy behaviours may not be listed.

The grades are:

***Budding* MathsCrafters**	Generates data and ideas
***Competent* MathsCrafters**	As above, and also pursues (some) ideas and critiques them
***Proficient* MathsCrafters**	As above, and also applies careful analysis, making logical arguments to support ideas, making progress towards being sure beyond all doubt

EVIDENCE

The main source of evidence for the assigned grade is the group's written account of their adventure sparked by the assessment problem/s.

Your observations of the students' behaviours during the problem session/s may also be used to decide on a grade.

THE ASSESSMENT PROBLEM SESSION/S

The job of the teacher during an assessment session is no different than in the other problems: to present the problem and assist the flow of the adventure.

Assisting the flow of the adventure requires you to do as little as possible, but as much as needed, to create the environment in which students have the chance to *have an idea*.

As with other problems, the assessment problem's adventure can span one or two sessions.

If using two sessions, written accounts should be collected by the teacher at the end of the first session and given back at the start of the second session, and then collected again at the end of that session. Students cannot bring any written materials into the second session to be copied into their written accounts.

When assessing the groups, the teacher should consider how much input they provided for a group to reach a point where they had their ideas.

SUPPORT TO THE AUSTRALIAN CURRICULUM: MATHEMATICS

As stated earlier, MathsCraft allows students to *do mathematics* with *facts, concepts, processes and associations* with which they are already *operational*. As a result, students are able to *explore, solve, create and understand*.

Accordingly, by engaging with MathsCraft, students will further develop their skills in the Australian Curriculum's Mathematics Proficiencies: **understanding**, **problem solving** and **reasoning**[5]. The Australian Curriculum's descriptions of these Mathematics proficiencies are given below.

By engaging with MathsCraft, students will also develop their skills with the Australian Curriculum's General Capabilities[6]. MathsCraft addresses *all* of the elements listed in **Critical and Creative Thinking** (description also given below), and a large number of those listed in **Personal and Social Capability**, **Numeracy** and **Literacy**. A detailed list of these General Capabilities, with those elements that are addressed by MathsCraft duly marked, is given in Appendix C, p.22.

Most current curricula include something like the Australian Curriculum's Proficiencies and General Capabilities.

MATHEMATICAL PROFICIENCY (V8.4): UNDERSTANDING

Students build a robust knowledge of adaptable and transferable mathematical concepts. They make connections between related concepts and progressively apply the familiar to develop new ideas. They develop an understanding of the relationship between the 'why' and the 'how' of mathematics. Students build understanding when they connect related ideas, when they represent concepts in different ways, when they identify commonalities and differences between aspects of content, when they describe their thinking mathematically and when they interpret mathematical information.

MATHEMATICAL PROFICIENCY (V8.4): PROBLEM SOLVING

Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively. Students formulate and solve problems when they use mathematics to represent unfamiliar or meaningful situations, when they design investigations and plan their approaches, when they apply their existing strategies to seek solutions, and when they verify that their answers are reasonable.

MATHEMATICAL PROFICIENCY (V8.4): REASONING

Students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising. Students are reasoning mathematically when they explain their thinking, when they deduce and justify strategies used and conclusions reached, when they adapt the known to the unknown, when they transfer learning from one context to another, when they prove that something is true or false, and when they compare and contrast related ideas and explain their choices.

GENERAL CAPABILITY (V8.4): CRITICAL AND CREATIVE THINKING

In the Australian Curriculum, students develop capability in critical and creative thinking as they learn to generate and evaluate knowledge, clarify concepts and ideas, seek possibilities, consider alternatives and solve problems. Critical and creative thinking involves students thinking broadly and deeply using skills, behaviours and dispositions such as reason, logic, resourcefulness, imagination and innovation in all learning areas at school and in their lives beyond school.

Thinking that is productive, purposeful and intentional is at the centre of effective learning. By applying a sequence of thinking skills, students develop an increasingly sophisticated understanding of the processes they can use whenever they encounter problems, unfamiliar information and new ideas. In addition, the progressive development of knowledge about thinking and the practice of using thinking strategies can increase students' motivation for, and management of, their own learning. They become more confident and autonomous problem-solvers and thinkers.

[5] https://www.australiancurriculum.edu.au/resources/mathematics-proficiencies/ (v8.4)
[6] https://www.australiancurriculum.edu.au/f-10-curriculum/general-capabilities/ (v8.4)

APPENDIX A: KEY CONCEPTS

Listed here are some key concepts underlying the MathsCraft approach.

The sections are organised according to some common themes, but can be read independently.

HOW TO HAVE IDEAS

TIME-LAG

We believe that a person must have had sufficient time to become skilled at using a (mathematical) tool in order to be able to wield it well while attempting to solve a problem. Partly this has to do with experience with the tool and knowing how to use it, and partly this has simply to do with how easily it will occur to you to use it.

WITH A GROUP OF STUDENTS

Of course, this implies that someone trying to solve a problem may need to already be skilled with a wide range of tools. This is why the MathsCraft Curriculum is *supplementary* to the standard curriculum – the standard curriculum is the place where students gain those tools.

As a rule of thumb, we consider a problem appropriate for students when the tools they need to access it are ones they learnt (in normal maths classes) some time ago, say around 3 years ago. Hence some MathsCraft problems may superficially seem quite simple for their intended audience.

But rest assured: there is plenty of beautiful, interesting, challenging mathematics that can be done with very simple tools.

DATA

When starting out on a problem, it can be useful to have a number of examples to look at. We call these *data*, using the term broadly. Generating data can allow you to get a feel for the structure of the problem, and through looking at the data you can notice things that may give insight. Not everything you notice will be significant, but the more you look the more you will see.

WITH A GROUP OF STUDENTS

There's sometimes an opportunity to have each person generate a small amount of data, and then pool it by having several of them write what they've got on the board. Data on the board is available for everyone to look at, so everyone has the opportunity to notice things.

Once conjectures are made that agree with all the available data, more data can be generated to test the conjecture. But keep in mind that even if the conjecture is supported by the new data, it still may be false (see "BAD" arguments).

OBSERVATIONS

To get to the bottom of a problem, a useful approach can be to simply look at what you've got so far (usually what you've got is Data) and see what you can observe. Observations are usually the starting point when trying to solve a problem – but we can't control what we observe! So the best approach is to simply observe anything you can. At this point, no fruit is too low — anything might be relevant. Whatever turns out not to be relevant can be put on The Shelf, or just put aside.

You can start by just trying to spot any patterns, but it can also be worthwhile to really try to think about what's going on: what the underlying rules are and how they might be producing the patterns you're seeing. This takes more mental effort but can be more fruitful.

Observations can themselves act as a form of data. Having a list of your observations written down in front of you, so you can look at them, may lead you to see connections between them.

Observation can also play a big role earlier in the process, it can be a way to find a problem to work on. A simple example is looking at a mathematical object, observing a property or two, and asking "how many other objects have those properties, and what else do they have in common?".

WITH A GROUP OF STUDENTS

As with data, there may be an opportunity for students to collect observations, and then share them with the room. Writing these on the board yourself will allow you to keep control of how clearly they're written, and avoid double-ups. Once written on the board they can provide fuel for more observations, or spark ideas or questions.

CONVERSATIONS AND DECISIONS

DEFINITIONS

In maths you will come across a lot of definitions. Each one has been decided on by someone, for a reason. And each one could have been defined differently.

Definitions are about communication: it's about making things precise so that when you talk to someone else you each know exactly what you're talking about.

But it's also about communicating with yourself. Definitions help you refer to a concept in your head, or on paper, so that you can think about it more easily (see also Notation).

Some definitions are set by convention — meaning that the mathematical community has broadly agreed on what that thing means. Following convention for such things makes communicating with others easier. But that doesn't mean those definitions are "correct", any more than the word "rock" is the correct word for a rock.

A lot of the time, the definition of any word, symbol or thing is up to you — choose a definition that seems like it will be most useful: one that makes things clear and easy to think about.

If things are hazy, you keep getting confused, or you're arguing too much with the people you're working with, it could be a sign that you need to go back and carefully define something.

WITH A GROUP OF STUDENTS

Once deep enough into the problem, there will often be arguments about what things mean. If there's disagreement on what the most useful definition will be, a simple way to avoid endless debate is to put it to a vote. If it later turns out it would have been better to define things differently, you can always go back and change your definition!

RULES

There are a lot of rules in maths. But these rules weren't handed down from on-high, and a lot of new maths is invented by asking "what would happen if I broke this rule?". Some of the simplest examples lead to very deep questions: For example, you can't divide by 0. But let's say you can — what would 1/0 mean? How is it related to infinity? Does arithmetic apply to infinity? If not, can we change the rules of arithmetic to make it apply?

Each piece of mathematics is based on a set of assumptions, definitions, and rules. But these can be changed — what's really important is to remain internally consistent. You can create a thing, make it follow some rules, and then find out what the consequences of those rules are.

So when faced with a set of rules in a problem, you can choose to stick to them, or you can choose to change them (see Think beyond). As long as you (and anyone you're working with) clearly know the rules that you are following, it's all good!

WITH A GROUP OF STUDENTS

Students seem to be used to the idea that they will be told what the rules are, and then they have to follow them.

Something fun to do (and often a new experience) is to let the students make the rules. This can be done by "starting loose" — being ambiguous about the rules. It can help to be not-obviously-ambiguous, because this will mean the students get to work and make assumptions (without realising it), rather than asking you to clarify. Then you can use the Data or Observations stages to expose these assumptions, have the students argue over the rules or definitions, and insist that the room come to an agreement, one way or another.

NOTATION

Notation can be a blessing and a curse. Firstly, standard notation for things is used for communication (for more on this see Definitions). But much like definitions, if you don't need to communicate with anyone else yet, you can invent your own notation.

Why would you want to do this, if notation already exists? Well, different notation can help you think about things differently. Sometimes, one aspect of a thing is not as important as another. You can choose or invent a notation that will hide the aspects you don't care about, and make obvious the ones you do. For example, if all that matters about a number is whether it's even or odd, consider a notation which masks the actual number but highlights evenness or oddness: represent numbers by 1/0, O/E, or so on.

You may need to change notation if the important aspects change.

WITH A GROUP OF STUDENTS

Invention is a huge part of doing maths. Students will invent all sorts of notation. As with definitions, there's usually no right or wrong, just pros and cons to each choice of notation. Having different students use the same notation can be helpful to them when sharing things, but is not necessary – things can be learnt from seeing someone else's way of doing things.

If a student uses a notation that you think will be helpful to the rest of the class, you can have them share it and point out the benefits of the notation. Students will naturally take it up if they think it will be useful to them.

"BAD" ARGUMENTS

ARE YOU SURE, SURE *BAD*?

There's being sure, and then there's being sure BAD — "Beyond All Doubt". This is MathsCraft slang for a proof: an argument that begins with assumptions and follows logical steps to reach a conclusion. If you have a BAD argument for something, you know that if the assumptions are true then the conclusion must be true.

The trick is finding the argument and knowing that it's a BAD one. We can easily convince ourselves that something is true when there's actually a hole in the argument – something we've overlooked or an error in logic. It's a good idea to practice scrutinizing your own argument and looking for flaws, but don't just rely on yourself: often the best way to find flaws is to get someone else to scrutinize it.

WITH A GROUP OF STUDENTS

The first step in allowing students to come up with BAD arguments is to not give them answers, and to not tell them if they're right or wrong. If they have an answer and they want to know if it's right, they should convince themselves and/or those they're working with. (In the world of maths research, there are no answers at the back of the book.)

Students' arguments will often be full of holes. This is completely natural, logical thinking takes practice! You can get good at spotting the holes and deciding how to respond. It's good if students can learn to spot the holes themselves, but sometimes they will need a pointer: something like "that's a nice argument, but I'm not quite convinced by this bit."

THE BOOK

The (in-)famous mathematician Paul Erdős was known for referring to what he simply called "The Book": the book in which God keeps the most elegant proofs of mathematical theorems. He once said "You don't have to believe in God, but you should believe in The Book."

Theorems can often be proved in multiple ways. Sometimes the proof is simply logically sound and correct (a BAD argument), which means the theorem is true. But sometimes the proof is also illuminating: it shows not only that the theorem is true but also why it's true. It lays bare the structure underlying the fact. These proofs are said to be more elegant. Usually the simpler the better.

There is usually little doubt when a proof is elegant. There is something aesthetically pleasing about it. When you create an elegant proof, you know it.

You can read more about The Book on Wikipedia.

THINK BEYOND

"Think beyond" is a phrase we use to suggest you try breaking the Rules, after having found a solution to the current problem. Often an easy way to do this is to change a number or parameter: if you've been working with a size 4 something-or-other, why not try size 5? Size 6? Size N?

This can be the same thing as "generalising". Thinking beyond, considering problems that are related but not quite the same, can show you how the one you've already considered fits into a bigger picture, and can often turn up new, unexpected maths.

Two related concepts are "Think before" and "Think within".

"Think before" goes in the opposite direction to "Think beyond": to use our earlier example, what about size 3? Size 2? Size 1? The small cases might not seem worth bothering with, but they can show how the rules work at the most basic level, which can give you new insights into the problem. And if the example is really simple, they won't take much of your time!

"Think within" is about taking another look at what you've done, and looking for another perspective, or connections that maybe you didn't see before.

WITH A GROUP OF STUDENTS

The key to these concepts is timing. Students who are not ready to move on from their current train of thought, won't.

THE SHELF

There are two general sorts of modes you can be in when doing maths — one is looking for a problem to solve, the other is trying to solve a problem. No matter which mode you're in, you're bound to have ideas about potential tangents. These can be ways to break the rules you're working with, or things you notice, or just a question you think of while working (see Rules, Observations, and Think beyond).

The more you're into a problem the more this will happen — you need to have played around a bit in order to think of things. But you may be in the second mode, trying to focus on achieving a task. This is where "the shelf" comes in.

Rather than getting distracted by the tangent, or alternatively saying "that doesn't follow the rules" or "that's irrelevant" and forgetting about it, you can put the idea *on the shelf* and maybe take it down later to investigate. It might lead nowhere or it might lead somewhere really interesting, and there's only one way to find out.

By the way, this is how mathematicians find new problems to work on.

WITH A GROUP OF STUDENTS

You could keep a list in the corner of the board of things that have come up that you don't want the room to be thinking about right now — visibly "put it on the shelf" so that the students know it hasn't been dismissed, and they can think about it later if they want. Some of these ideas might even come in handy!

You could also suggest students keep a list of these kinds of ideas in the back of their book.

APPENDIX B: MATHEMATICAL BEHAVIOURS

There are some things a mathematician (or young learner) might do when faced with an interesting problem to ponder. Below is a detailed, though by no means complete, description of such mathematical behaviours.

Developing these behaviours is the main goal of the MathsCraft Curriculum. The below description of these behaviours may help you to mentor-by-modelling and is to be used for assessment.

This list is **not** a recipe for solving problems, and it is **not** a checklist. Mathematical adventures are not linear, and it is possible that some behaviours will not be called upon during a given adventure.

These behaviours become more effective with experience, as the learner gains a feel for what is required. Such experience can be gained by working on many problems, alone or with other people. It is rarely gained by following a recipe.

> *"It is inherent in the nature of guidelines that they don't work if you take them too literally."*
>
> *Ian Stewart (foreword to How to solve it, George Pólya, 2nd ed, 1990)*

LIST OF MATHEMATICAL BEHAVIOURS

DOING THINGS TO HELP YOU HAVE AN IDEA

- Generating data — calculating
- Organising the data
- "Playing" around with the data, getting to know it
- Discussing it with someone
- Looking for patterns or connections
- Establishing a notation
- Deciding on a definition
- Trying a different notation
- Changing a definition
- Asking a question
- …

HAVING AN IDEA

- Observing something
- Thinking of a new way to write something
- Making an association between two things (often one being present in the data, but also a fact or idea that comes to mind)
- Noticing/sensing a pattern
- Making a prediction
- …

DOING SOMETHING WITH THAT IDEA

- Testing it against new data
- Sharing it with someone
- Extending it
- Discarding it
- Convincing someone of its validity
- Recognising gaps in logic
- Cleaning up the idea (working towards a precise statement)
- Forming an argument (working towards becoming sure, beyond all doubt)
- Writing a proof
- …

CREATING A NEW PROBLEM

- Looking before/within/beyond for other things of interest
- Changing, adding, or removing a rule
- Asking a new question
- Pondering other problems known to you and considering links
- …

APPENDIX C: AUSTRALIAN CURRICULUM: GENERAL CAPABILITIES

✓ *students can gain practice at this capability by participating in the MathsCraft Curriculum.* (*NB: the information below pertains to v8.4 of the Australian Curriculum*[7]*.*)

CRITICAL AND CREATIVE THINKING

Inquiring — identifying, exploring and organising information and ideas	
Pose questions	✓
Identify and clarify information and ideas	✓
Organise and process information	✓
Generating ideas, possibilities and actions	
Imagine possibilities and connect ideas	✓
Consider alternatives	✓
Seek solutions and put ideas into action	✓
Reflecting on thinking and processes	
Think about thinking (metacognition)	✓
Reflect on processes	✓
Transfer knowledge into new contexts	✓
Analysing, synthesising and evaluating reasoning and procedures	
Apply logic and reasoning	✓
Draw conclusions and design a course of action	✓
Evaluate procedures and outcomes	✓

NUMERACY

Estimating and calculating with whole numbers	
Understand and use numbers in context	✓
Estimate and calculate	✓
Use money	-
Recognising and using patterns and relationships	✓
Using fractions, decimals, percentages, ratios and rates	
Interpret proportional reasoning	✓
Apply proportional reasoning	✓
Using spatial reasoning	
Visualise 2D shapes and 3D objects	✓
Interpret maps and diagrams	-
Interpreting statistical information	
Interpret data displays	-
Interpret chance events	-
Using measurement	
Estimate and measure with metric units	-
Operate with clocks, calendars and timetables	-

[7] https://australiancurriculum.edu.au/f-10-curriculum/general-capabilities/

LITERACY

Comprehending texts through listening, reading and viewing	
Navigate, read and view learning area texts	-
Listen and respond to learning area texts	-
Interpret and analyse learning area texts	-
Composing texts through speaking, writing and creating	
Compose spoken, written, visual and multimodal learning area texts	✓
Use language to interact with others	✓
Deliver presentations	✓
Text knowledge	
Use knowledge of text structures	-
Use knowledge of text cohesion	-
Grammar knowledge	
Use knowledge of sentence structures	-
Use knowledge of words and word groups	-
Express opinion and point of view	✓
Word knowledge	
Understand learning area vocabulary	✓
Use spelling knowledge	-
Visual Knowledge	
Understand how visual elements create meaning	✓

PERSONAL AND SOCIAL CAPABILITY

Self-awareness	
Recognise emotions	-
Recognise personal qualities and achievements	-
Understand themselves as learners	✓
Develop reflective practice	✓
Self-management	
Express emotions appropriately	-
Develop self-discipline and set goals	✓
Work independently and show initiative	✓
Become confident, resilient and adaptable	✓
Social awareness	
Appreciate diverse perspectives	✓
Contribute to civil society	-
Understand relationships	-
Social management	
Communicate effectively	✓
Work collaboratively	✓
Make decisions	✓
Negotiate and resolve conflict	-
Develop leadership skills	-